TRINITY COLLEGE

Year 12 Mathematics Specialist 3/4
Test 5 2022
Weighting 7%

Calculator Assumed
Rates of Change and Differential Equations

STUDENT'S NAME Solutions [PRESSER]

DATE: Thursday 18 August **TIME:** 50 minutes **MARKS**: 51

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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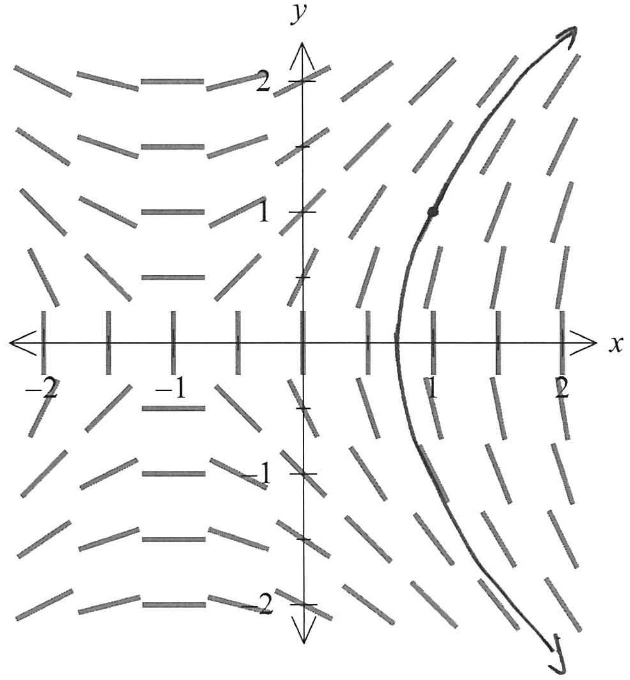
1. (10 marks)

The slope field given by $\frac{dy}{dx} = \frac{x+1}{y}$ is shown below.



(a) Determine the value of the slop field at the point

(i) $(1, 1)$ $\frac{dy}{dx} = \frac{2}{1} = 2$ ✓ ans [1]

(ii) $(-1, 0)$ $\frac{dy}{dx} = \frac{0}{0} = \text{indeterminant}$ [1]

✓ ans

(b) On the diagram above, draw the solution curve that contains the point $(1, 1)$. [2]

See above

✓ through (1,1)

✓ curve (bottom and top)

(c) Determine the equation of the solution curve that contains the point $(1,1)$. [3]

$$\frac{dy}{dx} = \frac{x+1}{y}$$

✓ separate

✓ integrate

✓ evaluate c and state eqn

$$\Rightarrow \int y\,dy = \int x+1\,dx$$

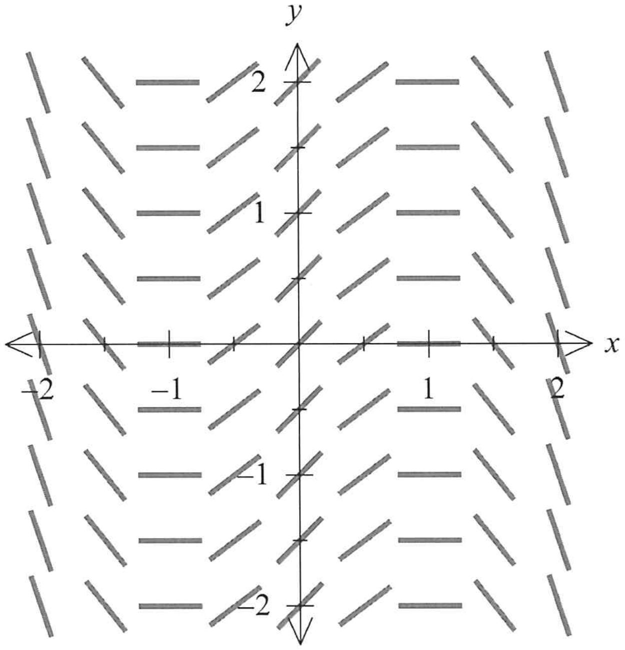
$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + x + c$$

$$\Rightarrow y^2 = x^2 + 2x + c_2$$

Sub in $(1,1)$, $c_2 = -2$

$$\text{So } y^2 = x^2 + 2x - 2$$

The slope field is adjusted. The new slope field generated is below.



(d) Determine the equation of the slope field given that at the point $(0,0)$ the slope is 1. [3]

Family of solution curves is order 3

$$\Rightarrow \frac{dy}{dx} = a(x+b)(x+c)$$

Slope is 0 at $x=-1$ and $x=1$ and slope is 1 at $(0,0)$

$$\Rightarrow 1 = a(0+1)(0-1)$$

$$\Rightarrow a = -1$$

✓ correct order

✓ uses slope 0

✓ ans

$$\text{So } \frac{dy}{dx} = -(x+1)(x-1)$$

2. (10 marks)

A population after t years is modelled by the differential equation

$$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

The initial value of the population was 100.

(a) Rewrite the logistic equation in the form $\frac{a}{1+be^{-ct}}$, clearly stating the values for a, b and c. [3]

$$\frac{dP}{dt} = \frac{6}{5}\,\frac{1}{4200}\,P(4200 - P)$$

✓ factorises

From this $kr = 1.2$, $k = 4200$

$$\Rightarrow \quad P = \frac{4200}{1 + \left(\frac{k - P_0}{P_0}\right)e^{-1.2t}}$$

✓ value a

With $P_0 = 100$

✓ value b and c

$$\Rightarrow \quad P = \frac{4200}{1 + 41e^{-1.2t}}$$

(b) Determine the rate of population growth when the population is 1500. [1]

$$\frac{dP}{dt} = 1.2(1500)\left(1 - \frac{1500}{4200}\right)$$

$= 1157.14$ /yr

✓ ans

(c) Determine how long it takes for the rate of population growth to reach a maximum. [3]

$$\frac{dP}{dt} = 1.2P - 1.2 \times \frac{1}{4200} P^2$$

✓ solves or states inflection ~~at~~ when $P = k/2$

$$\frac{d^2P}{dt^2} = 1.2 - 2.4 \times \frac{1}{4200} P$$

For a maximum rate, $\frac{d^2P}{dt^2} = 0$ ✓ state eqn to solve

$\Rightarrow P = 2100$

This is half the population and occurs when

$$2100 = \frac{4200}{1 + 41e^{-1.2t}}$$

$\Rightarrow t = 1.2 \ln 41$ ✓ value for t

$= 3.09$ yrs

(d) When the population reaches 4000, use the technique of increments to calculate the approximate change in population in the next month. [3]

$$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

1 month $\Rightarrow \Delta t = \frac{1}{12}$ ✓ $\Delta t = \frac{1}{12}$

So $\Delta P \approx \frac{dP}{dt} \times \Delta t$ ✓ uses increments

$= 1.2(4000)\left(1 - \frac{4000}{4200}\right) \times \frac{1}{12}$

$= 19.04$ /yr ✓ ans

3. (9 marks)

When a body, moving along the x-axis, has displacement x from the origin, its velocity v satisfies the equation

$$\frac{d}{dx}(v^2) = -18x$$

Given that $v = 4$ when $x = 1$,

(a) (i) Show that the velocity of the body is of the form $v = \sqrt{-9x^2 + c}$ [1]

$\Rightarrow \quad 2v \frac{dv}{dx} = -18x$ — ①

$\Rightarrow \quad \int 2v \, dv = \int -18x \, dx$

$\Rightarrow \quad v^2 = -9x^2 + c$ — ②

So $\quad v = \pm\sqrt{-9x^2 + c}$

(ii) Determine the velocity of the body when $x = 0$. [2]

Given $v = 4$ when $x = 1$

✓ evaluate c

$\Rightarrow \quad v^2 = -9x^2 + 25$

when $x = 0$, velocity is ± 5

✓ ans

(b) Determine the maximum displacement of the body from the origin. [3]

From eqn ②

$\Rightarrow \quad x^2 = \frac{25 - v^2}{9}$ ✓ eqn

For any value of v, x^2 has a max of $\frac{25}{9}$

$\therefore$ max displacement is $\frac{5}{3}$

✓ max x^2

✓ max x

(c) Determine the acceleration of the body when $x = 1$. [3]

$$a = v \frac{dv}{dx}$$

✓ expression for a

From eqn 1 we have $v \frac{dv}{dx} = -\frac{18x}{2}$

$\Rightarrow a = -9x$

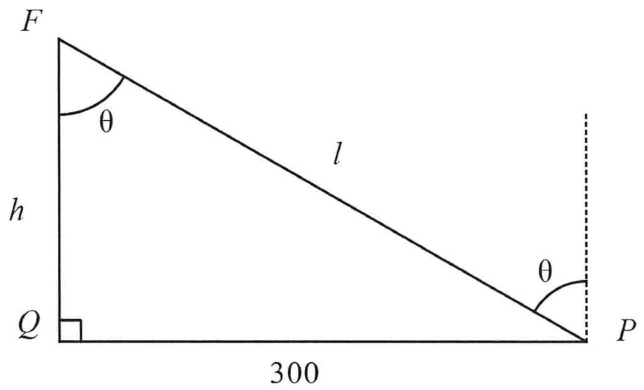
✓ uses information

So acceleration when $x = 1$ is -9

✓ ans

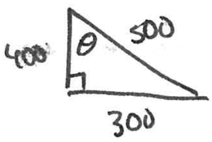
4. (14 marks)

Light from a flare F shines on a small plate P which lies on a horizontal plane. The intensity I of the light is directly proportional to $\frac{cos\theta}{l^2}$, where l is the length of the distance form F to P, and θ is the angle of incidence as shown in the diagram below.



At time t seconds the flare is h metres above the point Q on the plane, and P is 300 metres from Q. It can be shown that $I = k\ cos\theta\ sin^2\theta$ for some constant k.

(a) Evaluate k, given that $I = 72$ when $h = 400$. [2]



$\Rightarrow\ 72 = k \cdot \frac{4}{5} \cdot \left(\frac{3}{5}\right)^2$ ✓ sub

$\Rightarrow\ k = 250$ ✓ value

(b) Determine an expression for $\frac{dI}{d\theta}$. [2]

$I = k\cos\theta \sin^2\theta$

$\frac{dI}{d\theta} = k\cos\theta \cdot 2\sin\theta \cdot \cos\theta + \sin^2\theta \cdot k(-\sin\theta)$

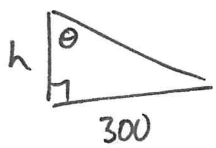
$= 250\sin\theta\left[2\cos^2\theta - \sin^2\theta\right]$

✓ product rule 1st
✓ product rule 2nd

(c) Determine, to the nearest metre, the height h at which the intensity I is the greatest. [2]

$\frac{dI}{d\theta} = 0 \Rightarrow \theta = 0,\ \sin^{-1}\left(\frac{\sqrt{6}}{3}\right) = 0.9553^r$

✓ θ



$\Rightarrow h = \frac{300}{\tan\theta}$

$= 212.13 \approx 212$ m

✓ height

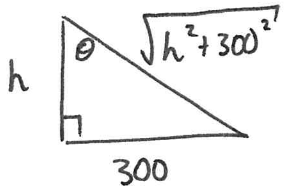
(d) Determine the maximum value of the intensity I. [1]

$I = 250 \cos 0.9553 \sin^2 0.9553$

$= 96.225$ ✓ ans

The flare is falling vertically at a constant rate of 5 metres per second.

(e) Show that $\frac{d\theta}{dt} = \frac{1}{60} sin^2\theta$. [4]



$$\tan\theta = 300h^{-1}$$

$$\Rightarrow \frac{1}{\cos^2\theta} \cdot \frac{d\theta}{dt} = \frac{-300}{h^2} \cdot \frac{dh}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta \times \frac{-300}{h^2} \times -5$$

$$= \frac{\cancel{h^2}}{h^2+300^2} \times \frac{1500}{\cancel{h^2}}$$

✓ eqn for θ & h

$$= \frac{1}{60} \cdot \frac{9000}{h^2+300^2}$$

✓ $\frac{d}{dt}$

$$= \frac{1}{60} \cdot \left(\frac{300}{\sqrt{h^2+300^2}}\right)^2$$

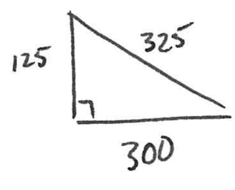
✓ expression for trig expression

$$= \frac{1}{60} \sin^2\theta$$

✓ factorisation

(f) Determine $\frac{dI}{dt}$ when the flare is 125 metres above the point Q. [3]

$$\frac{dI}{dt} = \frac{dI}{d\theta} \times \frac{d\theta}{dt}$$



$$= 250 \cdot \frac{300}{325}\left(2\left(\frac{125}{325}\right)^2 - \left(\frac{300}{325}\right)^2\right) \times \frac{1}{60}\left(\frac{300}{325}\right)^2$$

$$= -1.82 \text{ lumens/sec}$$

✓ chain rule

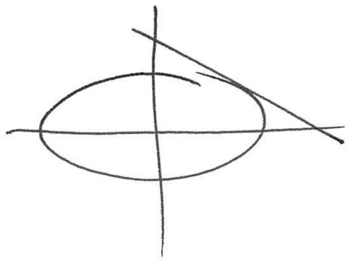
✓ values

✓ ans

5. (8 marks)

Consider the elliptical region $x^2 + 4y^2 = 5$.

(a) Determine the slope to the ellipse in the first quadrant when $x = 2$. [3]



$x^2 + 4y^2 = 5$

when $x = 2$

$\Rightarrow \quad 4 + 4y^2 = 5$

$\Rightarrow \quad y^2 = \frac{1}{4}$

$y = \pm \frac{1}{2}$

$\frac{d}{dx}(x^2 + 4y^2) = \frac{d}{dx}(5)$

$\Rightarrow \quad 2x + 8y \cdot \frac{dy}{dx} = 0$

✓ y-coordinate

$\Rightarrow \quad \frac{dy}{dx} = \frac{-x}{4y}$

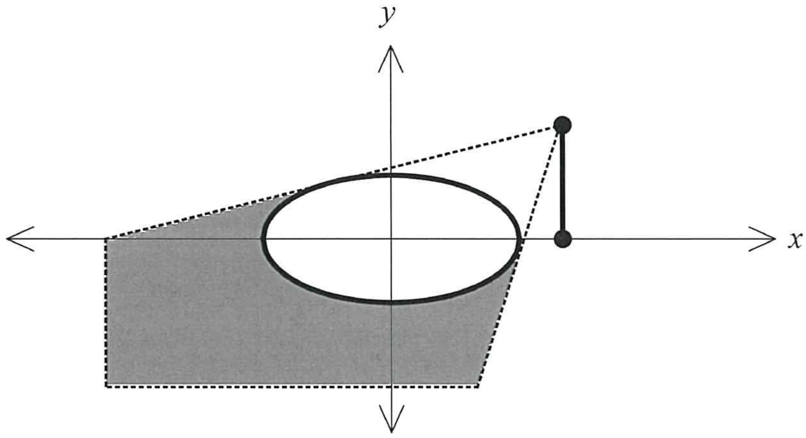
✓ derivative

✓ ans

$= \frac{-2}{4 \cdot \frac{1}{2}}$

$= -1$

A lamp is located at $x = 3$. A shadow is created by the ellipse as shown in the diagram below.



(b) If the point $(-5, 0)$ is on the edge of the shadow, determine how far above the x-axis the lamp is located. [5]

Eqn of tangent is $y - y_p = m(x - x_p)$

Subbing in point $(-5,0)$, tangent becomes

$\Rightarrow y = m(x+5)$ ✓ tangent

Point of intersection with the ellipse gives

$\Rightarrow x^2 + 4(mx + 5m)^2 = 5$ ✓ intersection

$\Rightarrow (4m^2+1)x^2 + 40m^2x + 100m^2 - 5 = 0$

For P.O.I to be tangent, $\Delta = 0$ ✓ quadratic

$\Rightarrow (40m^2)^2 - 4(4m^2+1)(100m^2-5) = 0$

$\Rightarrow m = \pm\frac{1}{4}$ ✓ m

$\therefore$ tangent line $y = \frac{1}{4}(x+5)$

when $x = 3$, $y = \frac{1}{4}(8)$

$= 2$ ✓ ans

$\therefore$ Height is 2 m